NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

V. Let ABC be $\triangle$ right-angled at C. Draw FD perpendicular to AB, meeting either leg produced. There are thus four similar right triangles.

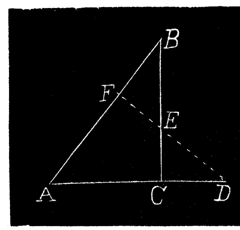

Fig. 5.

Letting $AC=b$, $AB=c$, $BC=A$, $CD=x$, $CE=y$, $AF=z$, $EB=a-y$, $FB=c-z$, $AD=b+x$, $FE=v$, $ED=w$, $FD=v+w$, we obtain the following proportions, with their resulting equations :

(1). $b : z :: c : b+x$. $\therefore b(b+x)=cz$........1.

(2). $b : z :: a : v+w$. $\therefore b(v+w)=az$......2.

(3). $c : b+x :: a : v+w$. $\therefore c(v+w)=a(b+x)$....................3.

(4). $b : y :: c : w$. $\therefore bw=cy$....................................4.

(5). $b : y :: a : x$. $\therefore bx=ay$....................................5.

(6). $c : w :: a : x$. $\therefore cx=aw$....................................6.

(7). $b : v :: c : a-y$. $\therefore b(a-y)=cv$..........................7.

(8). $b : v :: a : c-z$. $\therefore b(c-z)=av$..........................8.

(9). $c : a-y :: a : c-z$. $\therefore c(c-z)=a(a-y)$....................9.

(10). $z : y :: b+x : w$. $\therefore zw=y(b+x)$..........................10.

(11). $z : y :: v+w : x$. $\therefore xz=y(v+w)$..........................11.

(12). $b+x : w :: v+w : x$. $\therefore x(b+x)=w(v+w)$..................12.

(13). $z : v :: b+x : a-y$. $\therefore z(a-y)=v(b+x)$..................13.

(14). $z : v :: v+w : c-z$. $\therefore z(c-z)=v(v+w)$..................14.

(15). $b+x : a-y :: v+w : c-z$. $\therefore (c-z)(b+x)=(a-y)(v+w)$....15.

(16). $y : v :: w : a-y$. $\therefore y(a-y)=vw$..........................16.

(17). $y : v :: x : c-z$. $\therefore y(c-z)=vx$..........................17.

(18). $w : a-y :: x : c-z$. $\therefore w(c-z)=x(a-y)$..................18.

We are now to find combinations of the above equations from which the elements x, y, z, v, w, can be eliminated, thus leaving us the relation existing between a, b, and c.

It is evident that from no single equation, nor from any set of two equations, can the relation be determined.

There remains three possible cases of combinations to be considered :

1. When three of the elements x, y, z, v, w, are involved.
2. When four.
3. When five, or all.

First Case. Of this case there are $\frac{5.4.3}{\lfloor 3}=10$ possible combinations of three unknown elements : v, w, x ; v, w, y ; and so on.

Before taking up these in detail, we note that by inspection of the proportions, it easily may be seen that the following eighteen sets of equations each comprise dependent equations :

1, 2, 3 ; 4, 5, 6 ; 7, 8, 9 ; 10, 11, 12 ; 13, 14, 15 ; 16, 17, 18 ; 1, 4, 10 ; 1, 7, 13 ; 2, 5, 11 ; 2, 8, 14 ; 3, 6, 12 ; 3, 9, 15 ; 4, 7, 16 ; 5, 18, 17 ; 6, 9, 18 ; 10, 13, 16 ; 11, 14, 17 ; 12, 15, 18.

Hence, in our search for possible combinations, all such must be rejected as contain any of these sets.

There are three equations involving v, w, x: 3, 6, 12. But this combination must be rejected for the reason just given. For the same reason, or because there is wanting a sufficient number of equations involving the three unknown elements, the other nine combinations must be rejected, except the combination x, y, z, which elements are involved in equations 1, 5, 9. If we eliminate x, y, z from these equations, we obtain the desired relation, $c^2=a^2+b^2$.

It should be observed, in passing, that future combinations including 1, 5, 9, must also be rejected.

Second Case. Of this case there are $\frac{5.4.3.2}{\lfloor 4}=5$ possible combinations of four unknown elements ; and, besides, the exceptional combination, $v+w$, x, y, z, $v+w$ being regarded as a single unknown.

Before proceeding to investigate this case, it is necessary to call attention to sets of four dependent equations. Take, for example, the set 1, 2, 6, 12. From 1 and 2, 3 is obtained. But 3 with 6 and 12 gives a set of three dependent equations ; hence the set 1, 2, 6, 12 must be rejected. A little stndy of the eighteen sets given in Case 1, will disclose forty-five sets of four dependent equations.

The equations involving the unknown elements v, w, x, y, are 3, 4, 5, 6, 7, 12, 16. Out of these seven equations, there are $\frac{7.6.5.4}{\lfloor 4} = 35$ combinations, taking four at a time. Of these thirty-five sets, fourteen are to be rejected, for reasons previously stated. The remaining twenty-one sets, of which 7, 5, 4, 3, is a type, and to which the other twenty easily can be reduced, give, after the unknown elements have been eliminated, the desired relation between a, b, and c.

Similarly, we find twenty-one sets each of four equations, involving (v, w, x, z) and (v, w, y, z), and seventeen each involving (v, x, y, z), (w, x, y, z),

and $(v+w, x, y, z)$, thus making in all 114 proofs for this case.

THIRD CASE. Of this case, there are $\frac{18.17.16.15.14}{\lfloor 5}=8568$ sets of the eighteen equations, taking five at a time.

To determine how many of this number must be rejected, proceed as follows. Begin with the list of sets of dependent equations found in Case 1. Notice that there are $\frac{15.14}{\lfloor 2}=105$ sets of the eighteen equations taking five at a time, each containing equations 1, 2, 3; the same number containing equations 4, 5, 6; and so on, till we come to 1, 4, 10; for while there are 105 sets containing equations 1, 4, 10, three of them have already been counted out. So proceed, with the entire list of sets of dependent equations in Case 1, then with the set 1, 5, 9, following this with the sets of Case 2. We thus find that there are 3746 sets of five to be rejected, either because they contain sub-sets of dependent equations or sub-sets of equations from which the desired relation between a, b, c, is obtained.

One more class must be rejected: sets of five dependent equations. For example, 10, 9, 7, 6, 3, which is a type of all the others—72 in number—and from which the 72 can easily be deduced.

Deducting from 8568, 3746+73, we have remaining 4749 sets of five, from which can be derived the identity $c^2=a^2+b^2$.

$\therefore$ 1+114+4749=4864, the number of proofs by this method.

EXAMPLES :

1. $cv+cw-ax=ab$ .... 3.
$bw=cy$ .... 4.
$aw=cx$ .... 6.
$cv+by=ab$ .... 7.
4 in 6, $bx=ay$ .... 5.
4, 5 and 7 in 3, $ab-by+\frac{c^2y}{b}-\frac{a^2y}{b}=ab$. $\therefore c^2=a^2+b^2$.

2. $cz-bx=b^2$ .... 1.
$bv+bw-az=0$ .... 2.
$bw=cy$ .... 4.
$bx=ay$ .... 5.
$cv+by=ab$ .... 7.
1 in 2, $cv+cw-ax=ab$ .... 3.
4, 5, and 7 in 3, same as in 1st example.

VI. Let ABC be $\triangle$ right-angled at C. Produce AC to some point as D. Draw DF perpendicular to AB, produced, and meeting CB, produced.

Employing notation similar to that used in V., and proceeding somewhat in the same manner, we find that this method also yields a large number of proofs, in fact the same number that we found in V.

[To be Continued.]

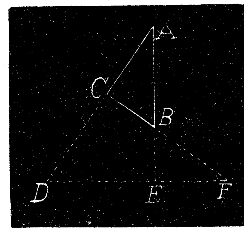

Fig. 6.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Prophsed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A, B, and C can walk at the rate of $a=3$, $b=4$, and $c=5$ miles, per hour. They start from Washington, at $m=1$, $n=2$, and $p=3$ o'clock, P. M., respectively. When B overtakes A, he is ordered (by A) back to C. When will B and C meet? Suppose B had ordered A back to C, when would A and C meet? In case all three continue walking ahead, at what time will they meet?

Solution by P. S. BERG, Larimore, North Dakota.

Since B gains 1 mile in 1 hour on A, to gain 3 miles will require 3 hours, or it will be 5 o'clock and 12 miles from starting point when B and A meet. C has traveled 10 miles. Since B and C travel 9 miles in 1 hour, they will travel 2 miles in $\frac{2}{9}$ hour, hence they will meet at $5\frac{2}{9}$ o'clock. Since A and C travel 8 miles in 1 hour, they will travel 2 miles in $\frac{1}{4}$ hour, hence they will meet at $5\frac{1}{4}$ o'clock.

In case all three continue walking ahead, as stated above A and B will meet at 5 o'clock. Since C gains 2 miles on A in 1 hour, to gain 6 miles will require 3 hours. Hence they will meet at 6 o'clock. Since C gains 1 mile on B in 1 hour, to gain 4 miles will require 4 hours. Hence it will be 7 o'clock when they meet.

Also solved by *B. F. YANNEY* and *H. C. WILKS*.

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{12}{25}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{6}{7}$ weeks longer than 660 sheep to eat up 9 acres.